A Geometric Approach to Diffraction Problems(Microlocal Analysis of Differential Equations)

Kataoka, Kiyomi

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A Geometric Approach to Diffraction Problems

Kiyômi Kataoka (片岡清臣)
Department of Math. University of Tokyo

In this paper, we introduce an improvement of the method and the results obtained by Kataoka-Tose [2]. As an important application to diffraction problems we give a new geometrical proof of Lebeau's result [3] on the second analyticity of solutions in the shadow. Further we obtain some theorems concerning microlocal mixed problems. These theorems are essentially in [2], but they are formulated entirely in a coordinate-invariant way for the future purpose.

Firstly we recall the formulation of boundary value problems by Kataoka-Tose [2]. For example, we consider the Dirichlet problem

$$\begin{cases} P(t,x,D_t,D_x)u(t,x) = 0 \quad (t>0) \\ u(+0,x) = 0 \end{cases} \tag{1}$$

where $(t,x)\in \mathbf{R}\times\mathbf{R}^n$, $D_t=\partial/\partial x$, $D_x=(\partial/\partial x_1,\dots,\partial/\partial x_n)$,

$$P(t,x,D_t,D_x)= D_t^2 + A_1(t,x,D_x)D_t + A_2(t,x,D_x)$$

(a second order C^ω-differential operator). Taking the canonical extension $\tilde{u}(t,x)$ of u, we can reduce (1) to the following degenerate problem

$$\begin{cases} t\cdot P(t,x,D_t,D_x)\tilde{u}(t,x)=0 \\ \mathrm{supp}(\tilde{u}) \subset \{t\geq 0\}. \end{cases}$$

That is, we can identify $\tilde{u}$ with a section of

$$\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\ \Gamma_{\{t\geq 0\}}(\mathcal{B}_{t,x})) \quad \text{for } \mathcal{M} = \mathcal{D}/\mathcal{D}tP. \tag{2}$$

On the other hand, noting that

$$\Gamma_{\{t\geq 0\}}(\mathcal{B}_{t,x}) = \mathcal{H}^n_{\{\mathrm{Im}z=0\}}(\Gamma_{\{t\geq 0\}}(\mathcal{B}_t\mathcal{O}_z)) = \mathbf{R}\Gamma_{\{\mathrm{Im}z=0\}}(\Gamma_{\{t\geq 0\}}(\mathcal{B}_t\mathcal{O}_z))[n],$$

we have $\mathbf{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\ \Gamma_{\{t\geq 0\}}(\mathcal{B}_{t,x})) = \mathbf{R}\Gamma_{\{\mathrm{Im}z=0\}}\mathbf{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\Gamma_{\{t\geq 0\}}(\mathcal{B}_t\mathcal{O}_z))[n]$. So we can consider $\tilde{u}(t,x)$ as a section of

$$\mathcal{H}^n_{\{\mathrm{Im}z=0\}}(\mathcal{F})$$

where

$$\mathcal{F} = \mathbf{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\ \Gamma_{\{t\geq 0\}}(\mathcal{B}_t\mathcal{O}_z)) = (\mathcal{O}^P_{\tilde{t},z} \cap \tilde{t}\mathcal{O}_{\tilde{t},z})|_W \otimes Y(t) \tag{3}$$

with $W=\{(\tilde{t},z)\in \mathbf{C}\times\mathbf{C}^n;\ \mathrm{Re}\tilde{t}\geq 0, \mathrm{Im}\tilde{t}=0\}$. Then we remark here that $\mathcal{F}$ has a natural extension $\tilde{\mathcal{F}}$ to the universal covering of $\{(\tilde{t},z)\in \mathbf{C}\times\mathbf{C}^n;\ \tilde{t}\neq 0\}$. Precisely, $\tilde{\mathcal{F}}$ is defined on the real monoidal space $^Y\tilde{X}=(X-Y)\cup S_YX$ with natural projection $\tau_{Y/X}$ onto X for $X=\mathbf{C}_{\tilde{t}}\times\mathbf{C}^n_z$ and the submanifold $Y=\{(\tilde{t},z)\in X;\ \tilde{t}=0\}$. If we write $\tilde{t}=re^{i\theta}$ with $(r,\theta)\in[0,+\infty)\times\mathbf{R}/2\pi\mathbf{Z}$, then

$$^Y\tilde{X} \cong [0,+\infty)_r\times S^1_\theta\times\mathbf{C}^n_z \xrightarrow{\tau_{Y/X}} (re^{i\theta},z)\in X, \tag{4}$$

and each stalk of $\tilde{\mathcal{F}}$ is given as follows :

$$\tilde{\mathcal{F}} = \begin{cases} \mathcal{O}^P_{\tilde{t},z} & \text{on } \{r\neq 0\}=X-Y, \\ (\mathcal{O}^P_{\tilde{t},z} \cap \tilde{t}\mathcal{O}_{\tilde{t},z})\cdot\left[\dfrac{\log\tilde{t}}{-2\pi i}\right] & \text{at } (0,\mathring{\theta},\mathring{z})\in S_YX, \end{cases} \tag{5}$$

where $\frac{\log \tilde{t}}{-2\pi i}$ is considered as an analytic function on

$\{\tilde{t}\in\mathbf{C};\ \tilde{t}\neq 0,\ \mathring{\theta}-\varepsilon<\arg\tilde{t}<\mathring{\theta}+2\pi+\varepsilon\}$, for some $\varepsilon>0$,

and $[\cdot]$ denotes the equivalence class modulo $\mathcal{O}_{\tilde{t},z}|_{(0,\mathring{z})}$. This is the reason why we introduce the following sheaf $\beta_Y(\mathcal{O}_X)$ instead of $\Gamma_{\{t\geq 0\}}(\mathcal{B}_t\mathcal{O}_z)$.

Definition 1. Let X,Y, and ${}^Y\tilde{X}$ be as above. Then we define a sheaf $\beta_Y(\mathcal{O}_X)$ on ${}^Y\tilde{X}\cong[0,+\infty)_r\times S^1_\theta\times\mathbf{C}^n_z$ by giving each stalk as follows :

$$\beta_Y(\mathcal{O}_X)|_{(\mathring{r},\mathring{\theta},\mathring{z})} = \begin{cases} \mathcal{O}_X|_{(\mathring{r}e^{i\mathring{\theta}},\mathring{z})} & \text{on } X-Y \text{ (i.e. } \mathring{r}>0), \\ \varinjlim_{\varepsilon\to+0}\{\tilde{\mathcal{O}}_X(W_\varepsilon)/\mathcal{O}_X(V_\varepsilon)\} & \text{on } S_YX \text{ (i.e. } \mathring{r}=0), \end{cases}$$

where $\tilde{\mathcal{O}}_X(W_\varepsilon)$ is the space of holomorphic functions on

$$W_\varepsilon = \{(\tilde{t},z)\in\widetilde{X-Y};\ 0<|\tilde{t}|<\varepsilon,\ |z-\mathring{z}|<\varepsilon,\ -\varepsilon<\arg\tilde{t}-\mathring{\theta}<2\pi+\varepsilon\}$$

with the universal covering space $\widetilde{X-Y}$ of X-Y, and

$$V_\varepsilon = \{(\tilde{t},z)\in X;\ |\tilde{t}|<\varepsilon,\ |z-\mathring{z}|<\varepsilon\ \}.$$

It is clear that this sheaf does not depend on a choice of coordinates and depends only on a pair, X and Y. Further, we introduce a notation $\alpha_Y(\mathcal{O}_X)$ for a sheaf on ${}^Y\tilde{X}$:

$$\alpha_Y(\mathcal{O}_X) = j_*(\mathcal{O}_X|_{X-Y})$$

with the imbedding $j:\ X-Y\longrightarrow {}^Y\tilde{X}$. It is easy to see that $\alpha_Y(\mathcal{O}_X)=\mathcal{O}_X$ on X-Y and that

$$\alpha_Y(\mathcal{O}_X)|_{(0,\mathring{\theta},\mathring{z})} = \varinjlim_{\varepsilon\to+0}\mathcal{O}_X(U_\varepsilon) \tag{6}$$

with

$$U_\varepsilon = \{(\tilde{t},z)\in X;\ 0<|\tilde{t}|<\varepsilon,\ |z-\mathring{z}|<\varepsilon,\ |\arg\tilde{t}-\mathring{\theta}\ |<\varepsilon\}.$$

Hence we have a natutral sheaf morphism

$$\kappa : \beta_Y(\mathcal{O}_X) \longrightarrow \alpha_Y(\mathcal{O}_X)$$

induced by

$$\begin{array}{rcl} \kappa_\varepsilon : \tilde{\mathcal{O}}_X(W_\varepsilon)/\mathcal{O}_X(V_\varepsilon) & \longrightarrow & \mathcal{O}_X(U_\varepsilon). \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [f(\tilde{t},z)] & \longmapsto & f(\tilde{t}e^{2\pi i},z)-f(\tilde{t},z) \end{array}$$

The following exact sequence is easily obtained, but the most important for the sheaf $\beta_Y(\mathcal{O}_X)$.

Proposition 2. We have an exact sequence of $\mathcal{D}_X$-modules

$$0 \longrightarrow \tau^{-1}_{Y/X}\mathcal{H}^1_Y(\mathcal{O}_X) \longrightarrow \beta_Y(\mathcal{O}_X) \xrightarrow{\kappa} \alpha_Y(\mathcal{O}_X) \longrightarrow 0 \qquad (7)$$

on ${}^Y\tilde{X}$.

By employing $\beta_Y(\mathcal{O}_X)$, we obtain another expression of $\tilde{\mathcal{F}}$ in (5):

$$\tilde{\mathcal{F}} = \mathbf{R}\mathcal{H}om_{\tau^{-1}_{Y/X}\mathcal{D}_X}(\tau^{-1}_{Y/X}\mathcal{M},\beta_Y(\mathcal{O}_X)) \qquad (8)$$

Then we have an estimate for the microsupport $SS(\tilde{\mathcal{F}})$ of $\tilde{\mathcal{F}}$, which is similar to our previous result in [2]. To simplify our situation, we embed ${}^Y\tilde{X}$ into a (2n+2)-dimensional C^ω-manifold Z as a submanifold with C^ω-boundary; that is,

$${}^Y\tilde{X} =\{(r,\theta,x,y)\in Z;r\geq 0\}\hookrightarrow Z=\mathbf{R}_r\times S^1_\theta\times\mathbf{R}^{2n}_{x,y} \xrightarrow{\tau} (re^{i\theta},x+iy)\in X. \qquad (9)$$

Further, every sheaf on ${}^Y\tilde{X}$ extends to Z as 0 on $\{r<0\}$.

Theorem 3. Let $\mathring{p}=(\mathring{r},\mathring{\theta},\mathring{x},\mathring{y};\mathring{r}^*dr+\mathring{\theta}^*d\theta+\mathrm{Re}(\mathring{\zeta}dz))$ be a point of T^*Z. Then, for the $\mathcal{D}_X$-module $\mathcal{M}=\mathcal{D}_X/\mathcal{D}_X\tilde{t}P$ defined in (2), $\mathring{p}$ does not belong to the microsupport of $\tilde{\mathcal{F}}=\mathbf{R}\mathcal{H}om_{\tau^{-1}\mathcal{D}_X}(\tau^{-1}\mathcal{M},\beta_Y(\mathcal{O}_X))$ if one of the following conditions is satisfied :

1) $\mathring{r}\neq 0$, and $\sigma(P)(\mathring{r}e^{i\mathring{\theta}},\mathring{z},e^{-i\mathring{\theta}}(\mathring{r}^*-i\frac{\mathring{\theta}^*}{\mathring{r}}),\mathring{\zeta})\neq 0$ when $\mathring{r}>0$,

2) $\mathring{r}=0$ and $\mathring{\theta}^*\neq 0$,

3) $\mathring{r}=\mathring{\theta}^*=0$ and the condition on roots of $\sigma(P)(0,\mathring{z},e^{-i\mathring{\theta}}(w+\mathring{r}^*),\mathring{\zeta})=0$ with respect to w: one root in $\{\mathrm{Re}\, w>0\}$ and the other one in $\{\mathrm{Re}\, w<0\}$.

Remark. In this theorem, we treat only a second order differential operator with the Dirichlet condition. However, the method of proof works also for any order operator with general boundary condition. In that case, condition 3) becomes more complicated because it includes a condition on the Lopatinski determinant (see [2]).

Proof. Set τ^* be a natural map

$$\tau^* : T^*X\times_X Z \longrightarrow T^*Z \tag{10}$$

induced by τ in (9). Then, in $\{\mathring{r}>0\}$, since τ is C^ω-diffeomorphic and $\beta_Y(\mathcal{O}_X)=\mathcal{O}_X$, $SS(\tilde{\mathcal{F}})=\tau^*(\mathrm{Char}(\mathcal{M}))=\{\sigma(P)\cdot(\tau^*)^{-1}=0\}$; that is, condition 1) implies $\mathring{p}\notin SS(\tilde{\mathcal{F}})$. To see case $\mathring{r}=0$, we rewrite $\beta_Y(\mathcal{O}_X)$ as follows:

$$\beta_Y(\mathcal{O}_X)=\{f(r,\theta,z)\in\Gamma_{\{r\geq 0\}}(\mathcal{B}_{r,\theta}\mathcal{O}_z);\ (rD_r+iD_\theta)f=0\}. \tag{11}$$

Hence the method of proof by Kataoka-Tose [2] works also in this case. We omit the details.

Let us apply this theorem to mixed problems with the Dirichlet condition. The first application is giving a new simple proof for Lebeau's theorem on diffraction [3]. The method of his proof is in constructing a parametrix by using Airy's function and second pseudodifferential operators. Our proof is based on some geometric arguments concerning the characteristic variety.

<u>Definition 4.</u> Let $P(t,x,D_t,D_x)$ be a differential operator with real principal symbol. Then, P is said to be strictly diffractive at a point $\mathring{p}=(0,\mathring{x};i\mathring{\eta}_t,i\mathring{\eta})\in i\mathring{T}^*(\mathbf{R}_t\times\mathbf{R}_x^n)$ from $\{t>0\}$ iff $\{t=0\}$ is non-characteristic for P and the following conditions (12),(13) are satisfied:

$$\sigma(P)(\mathring{p})=\{\sigma(P),t\}(\mathring{p})=0, \quad (d\sigma(P)\wedge dt\wedge\omega)(\mathring{p})\neq 0, \tag{12}$$

$$\{\sigma(P), \{\sigma(P),t\}\}\ (0,\mathring{x},\mathring{\eta}_t,\mathring{\eta}) > 0. \tag{13}$$

Here $\omega=\eta_t dt+\sum_{j=1}^{n}\eta_j dx_j$ and $\{f(t,x,\zeta_t,\zeta),g(t,x,\zeta_t,\zeta)\}=$

$$\frac{\partial f}{\partial\zeta_t}\frac{\partial g}{\partial t}-\frac{\partial g}{\partial\zeta_t}\frac{\partial f}{\partial t}+\sum_{j=1}^{n}\left(\frac{\partial f}{\partial\zeta_j}\frac{\partial g}{\partial x_j}-\frac{\partial g}{\partial\zeta_j}\frac{\partial f}{\partial x_j}\right).$$

In fact, under condition (12) the bicharacteristic ray passing through $\mathring{p}$ is tangent to $\{t=0\}$. Further, under (13) this ray is strictly tangent to $\{t=0\}$ from $\{t>0\}$. For a second order differential operator P, we may assume the form

$$P = D_t^2 + R(t,x,D_x) + \text{1-st order term} \tag{14}$$

after some coordinate transformation fixing $\{t=0\}$. Then the conditions (12), (13) reduce to the following :

$$\mathring{\eta}_t=\sigma(R)(0,\mathring{x},\mathring{\eta})=0, \quad (d\sigma(R(0,x,D_x))\wedge\sum_{j=1}^{n}\eta_j dx_j)\big|_{(\mathring{x};\mathring{\eta})}\neq 0,$$

and

$$\frac{\partial\sigma(R)}{\partial t}(0,\mathring{x},\mathring{\eta}) < 0.$$

In particular,

$$V = \{(x;i\eta);\ \sigma(R)(0,x,\eta)=0\} \tag{15}$$

is regular involutive in $iT^*\mathbf{R}^n_x$. We denote by $SS^2_V(f(x)) \subset T^*_V\tilde{V}$ the second spectrum of $f(x)$ along V. Here $\tilde{V}$ is the partial complexification of V with respect to natural Hamilton flows in V. To simplify our situation, we assume

$$V = \{\eta_1=0\}. \tag{16}$$

In fact, if we extends our operators to microdifferential operators, such a reduction of V is always possible under suitable quantized contact transformations. Then, $T^*_V\tilde{V}$ is expressed in local coordinates as

$$T^*_V\tilde{V} = \{(x;\eta'dx';ix^*_1dx_1)\}, \tag{17}$$

where $\eta'=(\eta_2,\ldots,\eta_n)$ and $x'=(x_2,\ldots,x_n)$. Further we have

$$(\overset{\circ}{x};\overset{\circ}{\eta}'dx';\pm idx_1)\notin SS^2_V(f) \xleftrightarrow[\text{equivalent}]{} f\in \mathcal{C}_{x',\mathcal{O}_{z_1}}|_{(\overset{\circ}{x};i\overset{\circ}{\eta}'dx')}. \tag{18}$$

That is, $SS^2_V(f)$ measures the gap between $\mathcal{C}_{x_1,x'}|_V$ and $\mathcal{C}_{x',\mathcal{O}_{z_1}}|_V$, where $\mathcal{C}_{x',\mathcal{O}_{z_1}}$ is the sheaf of microfunctions with holomorphic parameter $z_1=x_1+iy_1$. Under these preparations we state the theorem by Lebeau.

<u>Theorem 5.</u> (Lebeau [3]) Let $P=D_t^2 + R(t,x,D_x)$ + (lower order terms) be a differential operator as in (14) defined in a neighborhood of $(0,\overset{\circ}{x})\in \mathbf{R}_t\times\mathbf{R}^n_x$. We assume that

$$\sigma(R)|_{t=\eta_1=0} \equiv 0. \tag{19}$$

Let $\mathring{p}=(0,\mathring{x};i\mathring{\eta}dx)$ be a point of $i\mathring{T}^*(\mathbf{R}_t\times\mathbf{R}_x^n)\cap\{\eta_1=0\}$ such that

$$\frac{\partial\sigma(R)}{\partial t}(0,\mathring{x},\mathring{\eta})<0,\quad \frac{\partial\sigma(R)}{\partial\eta_1}(0,\mathring{x},\mathring{\eta})\neq 0. \tag{20'}$$

Let $u(t,x)$ be a hyperfunction solution to the problem

$$\begin{cases} Pu=0 & \text{in } \{0<t<\delta,\ |x-\mathring{x}|<\delta\}, \\ u(+0,x)=0 & \text{in } \{|x-\mathring{x}|<\delta\}, \\ SS(u)\cap\{0<t<\delta,|x-\mathring{x}|<\delta,|\eta-\mathring{\eta}|<\delta\} & \\ \qquad\subset\{(t,x;i\eta_t,i\eta);\ \eta_t\geq 0\ (\text{or } \eta_t\leq 0)\} & \end{cases} \tag{21}$$

for some $\delta>0$. Then we have $SS^2_{\{\eta_1=0\}}(D_tu(+0,x))=\emptyset$; that is, $D_tu(+0,x)$ depends holomorphically on x_1 near $(\mathring{x};i\mathring{\eta}'dx')$.

<u>Remark.</u> The last condition in (21) means that the solution u may have only shadow singularities; that is, $u(t,x)$ may have analytic singularities only on one portion of glancing rays divided by $\{t=0\}$.

<u>A sketch of our proof.</u> Let $X=\{(\tilde{t},z)\in\mathbf{C}\times\mathbf{C}^n\}\supset Y=\{(\tilde{t},z)\in X;\ \tilde{t}=0\}$ be complexifications of $\mathbf{R}_t\times\mathbf{R}_x^n$ and $\{0\}\times\mathbf{R}_x^n$, and put

$$\mathcal{M}=\mathcal{D}_X/\mathcal{D}_X\cdot\tilde{t}P(\tilde{t},z,D_{\tilde{t}},D_z). \tag{22}$$

Here $\tilde{t}$, z are the complexified variables for t,x. Further let $\tau:\ Z\hookrightarrow{}^{Y}\tilde{X}\ni(r,\theta,x+iy)\longmapsto(re^{i\theta},x+iy)\in X$ be the projection introduced in (9). Then we have

$$\tau^{-1}\Gamma_{\{t\geq 0\}}(\mathcal{B}_{t,x})|_{\theta=0}\simeq\mathbf{R}\Gamma_{\{\theta=0,\mathrm{Im}z=0\}}(\beta_Y(\mathcal{O}_X))[n+1] \tag{23}$$

by using Proposition 2. Hence the canonical extension $\tilde{u}$ of u belongs to $H^{n+1}(\mathbf{R}\Gamma_{\{\theta=0,\mathrm{Im}z=0\}}(\mathcal{F}))$ with a complex

$$\tilde{\mathcal{F}} = \mathbf{R}\mathcal{H}om_{\tau^{-1}\mathcal{D}_X}(\tau^{-1}\mathcal{M}, \beta_Y(\mathcal{O}_X)) \tag{24}$$

on Z. We denote by μ_* the microlocalization functor due to Kashiwara-Schapira [1], and consider the double microlocalization procedure $\mu_{\{\theta=\mathrm{Im}z_1=0\}}(\mu_{\{\mathrm{Im}z'=0\}}(\tilde{\mathcal{F}}))$ for $\tilde{\mathcal{F}}$. In fact, $\mu_{\{\mathrm{Im}z'=0\}}(\tilde{\mathcal{F}})$ is a complex of sheaves on

$$H_1 = T^*_{\{\mathrm{Im}z'=0\}}Z = \{(r,\theta,z_1,x';i\eta'dx') \in \mathbf{R}\times S^1\times\mathbf{C}\times iT^*\mathbf{R}^{n-1}\}, \tag{25}$$

and $\mu_{\{\theta=\mathrm{Im}z_1=0\}}(\mu_{\{\mathrm{Im}z'=0\}}(\tilde{\mathcal{F}}))$ on

$$H_2 = T^*_{\{\theta=\mathrm{Im}z_1=0\}}H_1 = \{(r,x_1,x';i\eta'dx';\theta^*d\theta+y_1^*dy_1)\}. \tag{26}$$

Let $\pi_1 : H_1 \longrightarrow Z\cap\{\mathrm{Im}z'=0\}$, $\pi_2 : H_2 \longrightarrow H_1\cap\{\theta=\mathrm{Im}z_1=0\}$ be natural projections. Then we have a canonical spectrum morphism:

$$sp^2 \ : \ \pi_2^{-1}\pi_1^{-1}\mathbf{R}\Gamma_{\{\theta=0,\mathrm{Im}z=0\}}(\tilde{\mathcal{F}}) \longrightarrow \mu_{\{\theta=\mathrm{Im}z_1=0\}}(\mu_{\{\mathrm{Im}z'=0\}}(\tilde{\mathcal{F}})). \tag{27}$$

On the other hand, as seen in (5), we know that $\tilde{\mathcal{F}}$ is just a sheaf on Z and that

$$\begin{array}{ccccc} \tilde{\mathcal{F}}|_{r=0} & \simeq & \tau^{-1}((\mathcal{O}_X^P\cap\tilde{t}\cdot\mathcal{O}_X)|_Y) & \xrightarrow{\sim} & \tau^{-1}(\mathcal{O}_Y) \\ & & \Psi & & \Psi \\ & & U(\tilde{t},z) & \longmapsto & D_{\tilde{t}}U(0,z) \end{array} \tag{28}$$

by Cauchy-Kowalevski's theorem. Therefore we also have a morphism (Neumann morphism) induced by (28):

$$\mu_{\{\theta=\mathrm{Im}z_1=0\}}(\mu_{\{\mathrm{Im}z'=0\}}(\tilde{\mathcal{F}}))|_{r=\theta^*=0}$$

$$\longrightarrow \mu_{\{\theta=\mathrm{Im}z_1=0\}}(\mu_{\{\mathrm{Im}z'=0\}}(\tau^{-1}\mathcal{O}_Y))|_{\theta^*=0}$$

$$= \tau^{-1}\mu_{\{\mathrm{Im}z_1=0\}}(\mu_{\{\mathrm{Im}z'=0\}}(\mathcal{O}_Y))|_{\theta^*=0}[-1]. \tag{29}$$

From the combination of morphisms (27) and (29) we obtain a splitting of the mapping

$$u(t,x) \longrightarrow [D_t u(+0,x)] \in \mathcal{C}^2_{\{\eta_1=0\}}$$

via $H^{n+1}(\mu_{\{\theta=\mathrm{Im}z_1=0\}}(\mu_{\{\mathrm{Im}z'=0\}}(\tilde{\mathcal{F}}))|_{r=\theta^*=0}$,where $\mathcal{C}^2_{\{\eta_1=0\}}$ is the sheaf of second microfunctions of x along $\{\eta_1=0\}$. Hence, in order to obtain the theorem we have only to show the following two facts: For $\sigma=-1$ (or $=1$),

(i) (The image of $\tilde{u}(t,x))\in H^{n+1}(\mathbf{R}\Gamma_{\{\sigma\theta^*\geq 0\}}(\mu_{\{\theta=\mathrm{Im}z_1=0\}}(\mu_{\{\mathrm{Im}z'=0\}}(\tilde{\mathcal{F}}))),$

(ii) $SS(\mu_{\{\theta=\mathrm{Im}z_1=0\}}(\mu_{\{\mathrm{Im}z'=0\}}(\tilde{\mathcal{F}})))\not\ni(0,\mathring{x};i\mathring{\eta}'dx';0d\theta\pm dy_1;\sigma d\theta^*)$.

Indeed, the fact (ii) induces the vanishing:

$$\mathbf{R}\Gamma_{\{\sigma\theta^*\geq 0\}}(\mu_{\{\theta=\mathrm{Im}z_1=0\}}(\mu_{\{\mathrm{Im}z'=0\}}(\tilde{\mathcal{F}})))|_{\theta^*=0} = 0.$$

The fact (i) is a corollary of Schapira's theorem:

<u>Theorem 6</u>. (Schapira [5]) Let u(t,x) be a hyperfunction satisfying the conditions in (21) except "u(+0,x)=0". Then, there exists a function $U(\tilde{t},z)$ holomorphic in a neighborhood of

$$\{(\tilde{t},z)\in\mathbf{C}\times\mathbf{C}^n;\ 0\leq\mathrm{Re}\tilde{t}<\varepsilon,\ 0\leq\mathrm{Im}\tilde{t}<\varepsilon\ \ (\text{or } 0\leq-\mathrm{Im}\tilde{t}<\varepsilon\ ),\ |z-\mathring{x}|<\varepsilon,\ \mathrm{Im}z\in\Gamma\}$$

with some $\varepsilon>0$ and $\Gamma=\{y\in\mathbf{R}^n;\langle y,\mathring{\eta}\rangle>\varepsilon\sqrt{|y|^2-\langle y,\mathring{\eta}\rangle^2}\}$ such that

$$SS\ ((U(t,z)Y(t))_{z=x+i0\Gamma}-\tilde{u}(t,x))\not\ni(0,\mathring{x};i\mathring{\eta}dx).$$

Theorem 4.1 of [5] is different from ours, but its proof essentially implies the above statement.

By using $U(\tilde{t},z)$ we can write $[\tilde{u}]$ as a boundary value of $U(\tilde{t},z)\frac{\log\tilde{t}}{-2\pi i}$ which is a section of $\beta_Y(\mathcal{O}_X)$ on

$$\{(r,\theta,z)\in Z;\ |r|<\varepsilon,\ |z-\mathring{x}|<\varepsilon,\ \mathrm{Im}z\in\Gamma,\ 0<\theta<\tfrac{\pi}{2}\ (\text{or } 0<-\theta<\tfrac{\pi}{2})\}\ .$$

Therefore we have fact (i) above.

The fact (ii) is a corollary of Theorem 3 and the estimation rules of microsupports under μ_*-functors due to Kashiwara--Schapira [1]. Indeed, to obtain (ii) we have only to show the following : There exist small positive constants ε, $\delta(\varepsilon)$, where $\delta(\varepsilon)$ is an increasing function of $\varepsilon\in(0,\varepsilon_0]$ and it tends to 0 as $\varepsilon\longrightarrow 0$. Then for any $(\varepsilon,\delta)\in\{0<\varepsilon<\varepsilon_0, 0<\delta<\delta(\varepsilon)\}$, any $\sigma=\pm1$ and any element of

$$\{(r,\eta_1^*,x;\eta';r^*,\theta^*,x^*,\eta'^*)\in\mathbf{R}\times\mathbf{R}\times\mathbf{R}^n\times\mathbf{R}^{n-1}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}^n\times\mathbf{R}^{n-1};\ 0\le r<\varepsilon_0,\ |\eta_1^*|<\varepsilon_0,\ |x-\mathring{x}|<\varepsilon_0,\ |\eta'-\mathring{\eta}'|<\varepsilon_0,\ |r^*|<\varepsilon_0,\ |\theta^*|<\varepsilon_0,\ |x^*|<\varepsilon_0,\ |\eta'^*|<\varepsilon_0\ \}$$

$SS(\tilde{F})$ does not contain the codirection

$$(r,-\varepsilon\sigma,x_1+i\varepsilon\eta_1^*,x'+i\varepsilon\delta\eta'^*;-\eta' dy'\pm\delta dy_1+\varepsilon\delta r^*dr+\delta\theta^*d\theta+\varepsilon\delta x^*dx)\in T^*L.$$

Further, by Theorem 3 we can reduce this condition on $SS(\tilde{F})$ to the following : One root with respect to w of

$$\sigma(P)(re^{-i\varepsilon\sigma},x_1+i\varepsilon\eta_1^*,x'+i\varepsilon\delta\eta'^*,e^{i\varepsilon\sigma}(w+\varepsilon\delta r^*),\varepsilon\delta x_1^*\mp i\delta,\varepsilon\delta x'^*+i\eta')=0$$

is in $\{\mathrm{Re}w>0\}$ and the other one in $\{\mathrm{Re}\,w<0\}$ for any ε, δ, σ, r, x, η', η_1^*, η'^*, r^*, x^* as above.

Since the roots w are given by

$$w=-\varepsilon\delta r^*\pm e^{-i\varepsilon\sigma}\sqrt{-\sigma(R)(re^{-i\varepsilon\sigma},x_1+i\varepsilon\eta_1^*,x'+i\varepsilon\delta\eta'^*,\varepsilon\delta x_1^*+i\delta\eta_1,\varepsilon\delta x'^*+i\eta')},$$

for each $\eta_1=\mp1$, it is sufficient to show that

$$|\sigma(R)| - \mathrm{Re}(e^{-2i\varepsilon\sigma}\sigma(R)) > 2\varepsilon^2\delta^2\varepsilon_0^2 \qquad (30)$$

On the other hand, by (20) we can write

$$-\sigma(R)(t,x,i\eta)=\sigma(R)(t,x,\eta)=-t\cdot A(t,x,\eta)+\eta_1\cdot B(t,x,\eta),$$

where A and B are real-valued analytic functions of real t,x,η satisfying

$$a=A(0,\mathring{x},\mathring{\eta})>0,\quad b=B(0,\mathring{x},\mathring{\eta})\neq 0.$$

Therefore,

$$E=-e^{-2i\varepsilon\sigma}\sigma(R)(re^{-i\varepsilon\sigma},x_1+i\varepsilon\eta_1^*,x'+i\varepsilon\delta\eta'^*,\varepsilon\delta x_1^*+i\delta\eta_1,\varepsilon\delta x'^*+i\eta')$$

$$=-re^{-3i\varepsilon\sigma}(a+\varepsilon_0C_1+i\varepsilon\varepsilon_0C_2)+\delta e^{-2i\varepsilon\sigma}(b\eta_1+\varepsilon_0C_3+i\varepsilon\varepsilon_0C_4)$$

$$=-ar(1+\varepsilon_0C_5)+b\eta_1\delta(1+\varepsilon_0C_6)$$

$$+i\sigma(ar\{\sin(3\varepsilon)+\varepsilon\varepsilon_0C_7\}-b\eta_1\delta\{\sin(2\varepsilon)+\varepsilon\varepsilon_0C_8\}). \tag{31}$$

Here $C_1,\ldots,C_8$ are real-valued analytic functions of $\varepsilon_0,\varepsilon,\delta,\ldots$ etc. which are bounded from above and below by constants independent of $\varepsilon_0,\varepsilon,\delta,\ldots$etc.. Hence, when $b\eta_1=-|b|$, we have

$$|E|+\mathrm{Re}E\geq|\mathrm{Im}E|^2/(2|E|)\geq\{(ar+|b|\delta)\varepsilon\}^2/\{(ar+|b|\delta)(3+6\varepsilon)\}$$

$$\geq|b|\varepsilon^2\delta/4, \tag{32}$$

for sufficiently small $\varepsilon_0>0$. Further, when $b\eta_1=|b|$, choose λ in $\{\frac{2}{3}<\lambda<1\}$. Then we have

$$|E|+\mathrm{Re}E\geq\mathrm{Re}E\geq(1-\lambda)|b|\delta/2\qquad\text{on }\{0\leq r\leq\lambda|b|\delta/a\}, \tag{33}$$

$$|E|+\mathrm{Re}E\geq|\mathrm{Im}E|^2/(2|E|)\geq\frac{(ar\{\sin(3\varepsilon)-(\sin(2\varepsilon)/\lambda)\}/2)^2}{ar(3+6\varepsilon)\{1+(1/\lambda)\}}$$

$$\geq\frac{1}{40}\lambda|b|\delta\{\sin(3\varepsilon)-\frac{\sin(2\varepsilon)}{\lambda}\}^2\qquad\text{on }\{r\geq\lambda|b|\delta/a\} \tag{34}$$

for sufficiently small $\varepsilon_0>0$. These estimates (32)~(34) lead to the condition (30) for sufficiently small $\varepsilon_0>0$. Thus the proof of Theorem 5 is completed.

Before treating microlocal Dirichlet problems, we introduce a theorem on the microsupport of $\mu_{\{\theta=0,\mathrm{Im}z=0\}}(\tilde{F})$ for $\tilde{F}=\mathbf{R}\mathcal{H}om_{\tau^{-1}\mathcal{D}_X}(\tau^{-1}\mathcal{M},\beta_Y(\mathcal{O}_X))$. Here,

$$\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X\tilde{t}\cdot P$$

and a second order operator

$$P = D_{\tilde{t}}^2 + R(\tilde{t},z,D_z)$$

defined in a neighborhood of $(0,\overset{\circ}{x})\in\mathbf{R}_t\times\mathbf{R}_x^n$.

<u>Theorem 7.</u> We assume that $\sigma(R)(t,x,i\eta)$ is real-valued for real t,x,η. Then we have the following estimates for the support and the microsupport of $\mathcal{G}=\mu_{\{\theta=0,\mathrm{Im}z=0\}}(\tilde{F})$ in a neighborhood of $(0,\overset{\circ}{x})$:

$$\mathrm{Supp}(\mathcal{G})\cap\{\eta\neq0\}\subset\{(r,x;\theta^*d\theta-\eta dy);\ r\geq0,\ (\theta^*)^2-r^2\sigma(R)(r,x,i\eta)=0\}$$
$$\cup\{r=\theta^*=0,\ \sigma(R)(0,x,i\eta)\geq0\},\qquad(35)$$

$$SS(\mathcal{G})\cap\{r=\theta^*=\sigma(R)=0,\eta\neq0\}\subset\{(0,x;0d\theta-\eta dy;r^*dr+x^*dx+\theta^{**}d\theta^*+\eta^*d\eta);$$
$$(\eta^*\cdot\frac{\partial}{\partial x}-x^*\cdot\frac{\partial}{\partial\eta})\sigma(R)(0,x,i\eta)=0\}.\qquad(36)$$

<u>Proof</u>. Since $\mathrm{Supp}(\mu_{\{\theta=0,\mathrm{Im}z=0\}}(\tilde{F}))\subset SS(\tilde{F})\cap T^*_{\{\theta=0,\mathrm{Im}z=0\}}L$, we can easily verify estimate (35) by Theorem 3. Further, to see (36) we must verify the following: For any codirection $\overset{\circ}{r}{}^*dr+\overset{\circ}{x}{}^*dx+\overset{\circ}{\theta}{}^{**}d\theta^*+\overset{\circ}{\eta}{}^*d\eta$ satisfying

$$a=(\overset{\circ}{\eta}{}^*\cdot\frac{\partial}{\partial x}-\overset{\circ}{x}{}^*\cdot\frac{\partial}{\partial\eta})\sigma(R)(0,\overset{\circ}{x},i\overset{\circ}{\eta})\neq0,\ \sigma(R)(0,\overset{\circ}{x},i\overset{\circ}{\eta})=0\qquad(37)$$

there exists a small constant $\varepsilon_0>0$ such that

$$\emptyset=SS(\tilde{F})\cap\{(r,-\varepsilon\theta^{**},x+i\varepsilon\eta^*;\varepsilon r^*dr+\theta^*d\theta+\mathrm{Re}((\varepsilon x^*+i\eta)dz))\in T^*L;$$
$$0<\varepsilon<\varepsilon_0,\ 0\leq r<\varepsilon_0,\ |x-\overset{\circ}{x}|<\varepsilon_0,\ |\eta-\overset{\circ}{\eta}|<\varepsilon_0,\ |r^*-\overset{\circ}{r}{}^*|<\varepsilon_0,\ |\theta^*|<\varepsilon_0,$$
$$|x^*-\overset{\circ}{x}{}^*|<\varepsilon_0,\ |\theta^{**}-\overset{\circ}{\theta}{}^{**}|<\varepsilon_0,\ |\eta^*-\overset{\circ}{\eta}{}^*|<\varepsilon_0\}.\qquad(38)$$

Therefore by the argument similar to one at (30) we have only to show the following inequality:

$$|\mathrm{Im}(e^{-2i\varepsilon\theta^{**}}\sigma(R))|^2 > 4\varepsilon^2(r^*)^2|\sigma(R)|, \qquad (39)$$

where $\sigma(R)=\sigma(R)(re^{-i\varepsilon\theta^{**}},x+i\varepsilon\eta^*,\varepsilon x^*+i\eta)$ with $\varepsilon,r,x,\eta,r^*,\theta^{**}$, x^*,η^* in (38). On the other hand, the assumption (37) implies the following equalities:

$$|\sigma(R)(re^{-i\varepsilon\theta^{**}},x+i\varepsilon\eta^*,\varepsilon x^*+i\eta)| = O(\varepsilon_0), \qquad (40)$$

$$\mathrm{Im}\{e^{-2i\varepsilon\theta^{**}}\sigma(R)(re^{-i\varepsilon\theta^{**}},x+i\varepsilon\eta^*,\varepsilon x^*+i\eta)\}$$
$$= \varepsilon(a+O(\varepsilon_0)). \qquad (41)$$

Clearly, the combination of (40) and (41) induces inequality (39) for a sufficiently small $\varepsilon_0>0$.

In order to combine $\mu_{\{\theta=0,\mathrm{Im}z=0\}}(\beta_Y(\mathcal{O}_X))$ with usual microlocal theory, we recall some complexes of relative microfunctions.

$$\mathcal{C}_{M_+|X} = \mu hom(\mathbf{Z}_{M_+},\mathcal{O}_X)[n+1], \qquad (42)$$

$$\mathcal{C}_{\Omega|X} = \mu hom(\mathbf{Z}_{\Omega},\mathcal{O}_X)[n+1], \qquad (43)$$

$$\mathring{\mathcal{C}}_{N|M_+} = \mathbf{R}\iota_!(\mathcal{C}_{\Omega|X}|_{T_N^*X}), \qquad (44)$$

where $M=\mathbf{R}_t\times\mathbf{R}_x^n \supset M_+=\{(t,x)\in M; t\geq 0\}\supset N=\{(t,x)\in M; t=0\}$, $\Omega=M_+-N$ and a natural map

$$\iota:\ T_N^*X \ni (0,x;wdt+i\eta dx)\longmapsto (x;i\eta)\in T_N^*Y. \qquad (45)$$

Here $\mathcal{C}_{M_+|X}$ and $\mathring{\mathcal{C}}_{N|M_+}$ are concentrated in degree 0, and we have the following facts :

$$\mathrm{Supp}(\mathcal{C}_{M_+|X})=\{(t,x;wdt+i\eta dx)\in T^*X; t\geq 0, \mathrm{Re}w\geq 0, t\cdot\mathrm{Re}w=0\},$$

$$\mathrm{Supp}(\mathcal{C}_{\Omega|X})=\{(t,x;wdt+i\eta dx)\in T^*X; t\geq 0, \mathrm{Re}w\leq 0, t\cdot\mathrm{Re}w=0\}.$$

Then, noting that $\mathrm{Supp}(\mathcal{C}_{M_+|X})$ and $\mathrm{Supp}(\mathcal{C}_{\Omega|X})$ are contained in $T_N^*X \cup T_M^*X$, we have natural maps

$$\tau^* : T_N^*X \cup T_M^*X \ni (t,x;wdt+i\eta dx) \longmapsto (t,x;-t\cdot \mathrm{Im}wd\theta-\eta dy) \in T_M^*Z, \tag{46}$$

$$d\tau : T_MZ \ni (r,x;\hat{\theta}\partial_\theta+\hat{y}\partial_y) \longmapsto (r,x;r\hat{\theta}\partial_{\mathrm{Im}\tilde{t}}+\hat{y}\partial_y) \in T_MX, \tag{47}$$

induced by τ: $Z \ni (r,\theta,z) \longmapsto (re^{i\theta},z) \in X$ at (9). Here M is identified with $\{(r,\theta,z) \in Z;\ \theta=0, \mathrm{Im}z=0\}$.

<u>Definition 8.</u> We define some complexes of sheaves on T_M^*Z as follows:

$$\mathcal{C}_{M_+|Z} = \mu_M(\beta_Y(\mathcal{O}_X))[n+1], \tag{48}$$

$$\mathcal{C}_{\Omega|Z} = \mu_M(\alpha_Y(\mathcal{O}_X))[n+1] = \mu_M(\mathbf{R}\Gamma_{X-Y}(\tau^{-1}\mathcal{O}_X))[n+1] \tag{49}$$

$$\overset{\circ}{\mathcal{C}}_{\Omega|Z} = \mathbf{R}(\tau^*)_!(\mathcal{C}_{\Omega|X}). \tag{50}$$

Easily to see, all these complexes coincide with $(\tau^*)_*(\mathcal{C}_M)$ in $\{r>0\}$. Further, since $\mathbf{R}(\tau^*)_!(\mathcal{C}_{\Omega|X})|_{\{r=0\}} = \mathbf{R}(\tau^*)_!(\mathcal{C}_{\Omega|X}|_{T_N^*X})$ and $\tau^*|_{T_N^*X} = \rho\cdot\iota$ with a natural closed injection

$$\rho : T_N^*Y \ni (x;i\eta dx) \longmapsto (0,x;0d\theta-\eta dy) \in T_M^*Z, \tag{51}$$

we obtain a relationship :

$$\overset{\circ}{\mathcal{C}}_{\Omega|Z}|_{\{r=0\}} = \rho_*(\overset{\circ}{\mathcal{C}}_{N|M_+}). \tag{52}$$

That is, $\overset{\circ}{\mathcal{C}}_{\Omega|Z}$ is concentrated in degree 0 and it combines the sheaf $\overset{\circ}{\mathcal{C}}_{N|M_+}$ of mild microfunctions with $\mathcal{C}_M|_{\{r>0\}}$ in a natural way.

<u>Remark</u>. From the exact sequence (7) we obtain a distinguished triangle as follows.

$$\rho_* \mu_N(\mathcal{H}^1_Y(\mathcal{O}_X))[n] \longrightarrow \mathcal{C}_{M_+|Z} \longrightarrow \mathcal{C}_{\Omega|Z} \xrightarrow{+1} . \tag{53}$$

Here the first one is concentrated in degree 0. Further we have the concentration in degree 0 for $\mathcal{C}_{\Omega|Z}$ at least when n=dimM-1=1. Therefore $\mathcal{C}_{M_+|Z}$ is concentrated in degree 0 at least when n=1.

It is well-known that there are two natural morphisms

$$\text{Trace} : \overset{\circ}{\mathcal{C}}_{N|M_+} \ni f(t,x) \longrightarrow f(+0,x) \in \mathcal{C}_N , \tag{54}$$

$$\text{ext} : \overset{\circ}{\mathcal{C}}_{N|M_+} \ni f(t,x) \longrightarrow \tilde{f}(t,x) \in \iota_*(\mathcal{C}_{M_+|X}|_{T^*_N X}). \tag{55}$$

Here $\tilde{f}(t,x)$ coincides roughly with $f(t,x)Y(t)$. Therefore, "Trace" and "ext" are defined also for $\overset{\circ}{\mathcal{C}}_{\Omega|Z}$ as

$$\text{Trace} : \rho^{-1}(\overset{\circ}{\mathcal{C}}_{\Omega|Z}|_{\{r=0\}}) \longrightarrow \mathcal{C}_N, \tag{56}$$

$$\text{ext} : \overset{\circ}{\mathcal{C}}_{\Omega|Z} \longrightarrow (\tau^*)_*(\mathcal{C}_{M_+|X}). \tag{57}$$

In particular, "ext" is a sheaf imbedding. Unfortunately, "ext" is not $\mathcal{D}_X$-linear, but there is a formula for differentiation; for example,

$$D_t(\text{ext}(f)) = \text{ext}(D_t f) + \text{Trace}(f)\cdot\delta(t). \tag{58}$$

Therefore it is convenient to introduce a $\mathcal{D}_X$-submodule of $(\tau^*)_*(\mathcal{C}_{M_+|X})$ including $\text{ext}(\overset{\circ}{\mathcal{C}}_{\Omega|Z})$.

<u>Proposition 9.</u> Put a complex on $T^*_M Z$:

$$\mathcal{C}_{N|Z} = \rho_*\mu_N(\mathbf{R}\Gamma_Y(\mathcal{O}_X))[n+1]. \tag{59}$$

Then $\mathcal{C}_{N|Z}$ is concentrated in degree 0 and it is a $\mathcal{D}_X$-submodule of $(\tau^*)_*(\mathcal{C}_{M_+|X})$. Further the sum

$$\overset{\circ}{\mathcal{C}}_{M_+|Z} = \text{ext}(\overset{\circ}{\mathcal{C}}_{\Omega|Z}) + \mathcal{C}_{N|Z} \tag{60}$$

is a direct sum and it is a $\mathcal{D}_X$-submodule of $(\tau^*)_*(\mathcal{C}_{M_+|X})$. In particular, we have an exact sequence of $\mathcal{D}_X$-modules on T_M^*Z:

$$0 \longrightarrow \mathcal{C}_{N|Z} \longrightarrow \overset{\circ}{\mathcal{C}}_{M_+|Z} \longrightarrow \overset{\circ}{\mathcal{C}}_{\Omega|Z} \longrightarrow 0. \qquad (61)$$

Proof. The concentration of $\mathcal{C}_{N|Z}$ follows from the well--known vanishing theorem due to Kashiwara on relative cohomology groups for $\mathcal{O}_X$. Hence we have an exact sequence of sheaves on N:

$$0 \longrightarrow \mathcal{H}_Y^1(\mathcal{O}_X)|_N \longrightarrow \Gamma_N(\mathcal{B}_M) \longrightarrow \overset{\circ}{\pi}_{M/Z*}(\mathcal{C}_{N|Z}) \longrightarrow 0. \qquad (62)$$

Here $\Gamma_N(\mathcal{B}_M)=\mathcal{C}_{N|Z}|_{T_Z^*Z}$ and $\mathcal{H}_Y^1(\mathcal{O}_X)|_N=\Gamma_{T_Z^*Z}(\mathcal{C}_{N|Z})$. On the other hand there is a natural sheaf-morphism :

$$\mathcal{C}_{N|Z}=\rho_*\mu_N(\mathbf{R}\Gamma_Y(\mathcal{O}_X))[n+1] \longrightarrow \mathbf{R}(\tau^*)_*(\mu_N(\mathcal{O}_X)[n+1])=(\tau^*)_*(\mathcal{C}_{N|X}). \qquad (63)$$

Therefore we have natural $\mathcal{D}_X$-morphisms:

$$\mathcal{C}_{N|Z} \longrightarrow (\tau^*)_*(\mathcal{C}_{N|X}) \longrightarrow (\tau^*)_*(\mathcal{C}_{M_+|X}) \longrightarrow (\tau^*)_*(\mathcal{C}_M). \qquad (64)$$

Since $\mathcal{H}_Y^1(\mathcal{O}_X)|_N=\{f(x)\in\Gamma_N(\mathcal{B}_M);SS(f)\subset iT_N^*M\}$, by using (62) and (64) we know that the natural morphism

$$\overset{\circ}{\pi}_{M/Z*}(\mathcal{C}_{N|Z}) \longrightarrow \overset{\circ}{\pi}_*(\mathcal{C}_M) \qquad (65)$$

is injective. Hence in order to prove the injectivity of the morphism : $\mathcal{C}_{N|Z} \longrightarrow (\tau^*)_*(\mathcal{C}_{M_+|X})$, we have only to construct a splitting of identity : $\mathcal{C}_{N|Z} \longrightarrow \mathcal{C}_{N|Z}$ by two integral transformations Φ_1, Φ_2 such that

$$\mathcal{C}_{N|Z}|_{\{|\eta|=1\}} \xrightarrow{\Phi_1} \overset{\circ}{\pi}'_*(\mathcal{C}_{N'|Z'}) \xrightarrow{\Phi_2} \mathcal{C}_{N|Z}|_{\{|\eta|=1\}}. \qquad (66)$$

Here $N'=N\times S^{n-1} \simeq \{(0,x;0d\theta-\eta dy)\in T_M^*Z;|\eta|=1\}$, and $Z'=Z\times(S^{n-1})^{\mathbf{C}}$.

Indeed, we can employ the kernel functions similar to those used in the proof of the flabbiness of microfunctions due to M. Kashiwara. We omit the details. Moreover note the following commutative diagram:

$$\begin{tikzcd}[column sep=small] \overset{\circ}{\mathcal{C}}_{\Omega|Z}=H^0(\mathbf{R}(\tau^*)_!(\mathcal{C}_{\Omega|X})) \arrow[rr, "\varphi_1"] \arrow[dr, "\mathrm{ext}"'] & & (\tau^*)_*(\mathcal{C}_{M_+|X})/(\tau^*)_*(\mathcal{C}_{N|X}) \\ & (\tau^*)_*(\mathcal{C}_{M_+|X}) \arrow[ur, "\varphi_2"'] & \end{tikzcd} \tag{67}$$

Here φ_1, φ_2 are natural morphisms and φ_1 is injective. Hence, considering morphisms (64) we obtain the injectivity of the morphism

$$\overset{\circ}{\mathcal{C}}_{M_+|Z}=\mathrm{ext}(\overset{\circ}{\mathcal{C}}_{\Omega|Z})+\mathcal{C}_{N|Z} \longrightarrow (\tau^*)_*(\mathcal{C}_{M_+|X})$$

from that of the morphism: $\mathcal{C}_{N|Z} \longrightarrow (\tau^*)_*(\mathcal{C}_{M_+|X})$. Simultaniously we proved that $\mathrm{ext}(\overset{\circ}{\mathcal{C}}_{\Omega|Z})\cap \mathcal{C}_{N|Z}=0$. The closedness of $\overset{\circ}{\mathcal{C}}_{M_+|Z}$ under $\mathcal{D}_X$-operation is directly derived from Lemma 10. This completes the proof.

We recall here Fourier-Sato transforms defined by Kashiwara--Schapira in [1]. Let p_1, p_2 be projections in

$$T_MZ \xleftarrow{p_1} T_MZ \underset{M}{\times} T^*_MZ \xrightarrow{p_2} T^*_MZ,$$

and $P_\pm$ be closed subsets of $T_MZ\times_M T^*_MZ$ given by

$$P_\pm = \{(r,x;\hat{\theta}\partial_\theta+\hat{y}\partial_y;\theta^*d\theta+y^*dy);\ \pm(\hat{\theta}\theta^*+\hat{y}y^*)\geq 0\}. \tag{68}$$

For $F\in \mathrm{Ob}(D^+_{\mathrm{conic}}(T_MZ))$, $G\in\mathrm{Ob}(D^+_{\mathrm{conic}}(T^*_MZ))$ we set

$$\begin{aligned} F^\wedge &= \mathbf{R}p_{2*}(\mathbf{R}\Gamma_{P_+}(p_1^{-1}F)),\\ G^\vee &= \mathbf{R}p_{1*}(\mathbf{R}\Gamma_{P_-}(p_2^{-1}G))[n+1]. \end{aligned} \tag{69}$$

$F^\wedge$ is called the Fourier-Sato transform of F and $G^\vee$ the inverse

Fourier-Sato transform of G, respectively. Indeed "$\vee$" is the inverse of "^". Here we omitted the orientation sheaf. Since $\mu_M(F)=\nu_M(F)^\wedge$ for a $F\in \mathrm{Ob}(D^+(Z))$, we may study $\nu_M(F)$ instead of $\mu_M(F)$. We set open subsets $D_\varepsilon(\mathring{x},\hat{y})$, $D_\varepsilon^+(\mathring{x},\hat{y})$ of $\mathbf{C}^{n+1}$ for any $\varepsilon>0$ and any $(0,\mathring{x};\hat{\theta}\partial_\theta+\hat{y}\partial_y)\in T_M Z$:

$$D_\varepsilon(\mathring{x},\hat{y})=\{(\tilde{\tau},x+i\lambda y)\in \mathbf{C}^{n+1};\ |\tilde{\tau}|+|x-\mathring{x}|<\varepsilon,\ 0<\lambda<\varepsilon,\ |y-\hat{y}|<\varepsilon\}, \tag{70}$$

$$D_\varepsilon^+(\mathring{x},\hat{y})=\{(\tilde{\tau},x+i\lambda y)\in \mathbf{C}^{n+1};\ |\tilde{\tau}|+|x-\mathring{x}|<\varepsilon,\ 0<\lambda<\varepsilon,\ |y-\hat{y}|<\varepsilon,\ (-\mathrm{Re}\tilde{\tau})_+ + |\mathrm{Im}\tilde{\tau}|<\lambda\}. \tag{71}$$

<u>Lemma 10.</u> For any $p_0=(0,\mathring{x};\hat{\theta}\partial_\theta+\hat{y}\partial_y)\in T_M Z$ we have

$$H^q((\mathcal{C}_{N|Z})^\vee[-n-1])_{p_0} = \varinjlim_{\varepsilon\to+0} H_Y^{q+1}(D_\varepsilon(\mathring{x},\hat{y}),\mathcal{O}_X) \quad (\forall q), \tag{72}$$

$$H^q((\mathring{\mathcal{C}}_{\Omega|Z})^\vee[-n-1])_{p_0} = \varinjlim_{\varepsilon\to+0} H^q(D_\varepsilon^+(\mathring{x},\hat{y}),\mathcal{O}_X) \quad (\forall q). \tag{73}$$

In particular, $(\mathcal{C}_{N|Z})^\vee[-n-1]$ and $(\mathring{\mathcal{C}}_{\Omega|Z})^\vee[-n-1]$ are concentrated in degree 0 and we have a $\mathcal{D}_X$-closed expression for $(\mathring{\mathcal{C}}_{M_+|Z})^\vee$:

$$(\mathring{\mathcal{C}}_{M_+|Z})^\vee[-n-1]|_{p_0} = \varinjlim_{\varepsilon\to+0} \{\mathcal{O}_X(D_\varepsilon^+(\mathring{x},\hat{y}))Y(t)+H_Y^1(D_\varepsilon(\mathring{x},\hat{y}),\mathcal{O}_X)\}. \tag{74}$$

<u>Proof.</u> (72) and (73) are derived from general theorems in [1] concerning microlocalization. Further the concentration of cohomology groups follows from the convexity of D_ε and D_ε^+ with small $\varepsilon>0$.

<u>Proposition 11.</u> There exist natural $\mathcal{D}_X$-morphisms of complexes on T_M^*Z:

$$\mathring{\mathcal{C}}_{M_+|Z} \xrightarrow{\ \psi_1\ } \mathcal{C}_{M_+|Z} \xrightarrow{\ \psi_2\ } (\tau^*)_*(\mathcal{C}_{M_+|X}) \tag{75}$$

such that $\psi_2 \cdot \psi_1$ gives the original sheaf imbedding of $\overset{\circ}{C}_{M_+}|Z$ to $(\tau^*)_*(C_{M_+}|X)$.

Proof. It is sufficient to see that the natural $\mathcal{D}_X$-morphism

$$(\overset{\circ}{C}_{M_+}|Z)^\vee[-n-1] \longrightarrow ((\tau^*)_*(C_{M_+}|X))^\vee[-n-1] \tag{76}$$

splits through $(C_{M_+}|Z)^\vee[-n-1]=\nu_M(\beta_Y(\mathcal{O}_X))$. Since all the complexes in (76) are equal to each other on $\{r\neq 0\}$, we have only to construct such a splitting in a neighborhood of $\{r=0\}$. Note that $((\tau^*)_*(C_{M_+}|X))^\vee[-n-1]=\mathbf{R}p_{1*}(\mathbf{R}\Gamma_{P_-}(p_2^{-1}(\tau^*)_*(C_{M_+}|X))$ is concentrated in degree ≥ 0, and that

$$\begin{aligned} H^0((\tau^*)_*(C_{M_+}|X)^\vee[-n-1])_{p_0} &= p_{1*}(\Gamma_{P_-}(p_2^{-1}(\tau^*)_*(C_{M_+}|X)))|_{p_0} \\ &= \{f(t,x)\in\Gamma_{M_+}(\mathcal{B}_M)|_{(0,\overset{\circ}{x})};SS(f)\subset\{(t,x;i\eta_t dt+i\eta dx);t\hat{\theta}\eta_t+\hat{y}\eta>0\} \\ &\qquad \cup\{t\eta_t=0,\ \eta=0\}\ \} \end{aligned} \tag{77}$$

for $p_0=(0,\overset{\circ}{x};\hat{\theta}\partial_\theta+\hat{y}\partial_y)\in T_M Z$. On the other hand,

$$H^q(\nu_M(\beta_Y(\mathcal{O}_X)))_{p_0} = \varinjlim_{\varepsilon\to +0} H^q(W_\varepsilon(p_0),\beta_Y(\mathcal{O}_X)) \quad (\forall q), \tag{78}$$

with

$$\begin{aligned} W_\varepsilon(p_0)=\{(r,\lambda\theta,x+i\lambda y)\in Z;0\leq r<\varepsilon,|x-\overset{\circ}{x}|<\varepsilon,0<\lambda<\varepsilon, \\ |\theta-\hat{\theta}|+|y-\hat{y}|<\varepsilon\}. \end{aligned} \tag{79}$$

Since $\tau(W_\varepsilon(p_0))\setminus Y$ is biholomorphic to a convex domain

$$\{(\alpha+i\lambda\theta,x+i\lambda y)\in\mathbf{C}\times\mathbf{C}^n;\alpha<\log\varepsilon,|x-\overset{\circ}{x}|<\varepsilon,0<\lambda<\varepsilon,|\theta-\hat{\theta}|+|y-\hat{y}|<\varepsilon\ \},$$

we obtain the concentration of $\nu_M(\alpha_Y(\mathcal{O}_X))$ in degree 0. Then by using exact sequence (7) we obtain also the concentration of

$\nu_M(\beta_Y(\mathcal{O}_X))$ in degree 0. Moreover note that we have the following identification

$$\Gamma(W_\varepsilon(p_0),\beta_Y(\mathcal{O}_X))=\{f(r,\theta,x,y)\in\Gamma_{\{r\geq 0\}}(W_\varepsilon(p_0),\mathcal{B}_{r,\theta}\mathcal{O}_Z);$$
$$(rD_r+iD_\theta)f=0\ \}. \qquad (80)$$

Let $f(r,\theta,x,y)$ be any section as above. Then a hyperfunction

$$g(r,x,\lambda):= f(r,\lambda\hat{\theta},x,\lambda\hat{y}) \qquad (81)$$

$$\in\Gamma_{\{r\geq 0\}}(\{(r,x,\lambda)\in\mathbf{R}\times\mathbf{R}^n\times\mathbf{R};r<\varepsilon,|x-\mathring{x}|<\varepsilon,0<\lambda<\varepsilon\},\ \mathcal{B}_{r,x,\lambda})$$

is well-defined and it satisfies an equation

$$D_\lambda g = \hat{\theta}D_\theta f + \hat{y}D_y f =(i\hat{\theta}rD_r + i\hat{y}D_x)g. \qquad (82)$$

In particular the boundary value $g(r,x,+0)$ is well-defined as a hyperfunction in $\Gamma_{\{r\geq 0\}}(\{r<\varepsilon,|x-\mathring{x}|<\varepsilon\},\ \mathcal{B}_{r,x})$ with an estimation

$$SS(g(r,x,+0))\subset\{(r,x;i\eta_r dr+i\eta dx);r\hat{\theta}\eta_r+\hat{y}\eta\geq 0\ \}.$$

It is easy to see that the boundary value $g(r,x,+0)$ does not depend on the choice of half line $\{y=\lambda\hat{y},\ \theta=\lambda\hat{\theta},\ 0<\lambda<\varepsilon\}$ when $(\hat{y},\hat{\theta})$ moves. Consequently we have an estimation of $SS(g(r,x,+0))$ as in (77). Therefore we obtain a natural $\mathcal{D}_X$-morphism for complexes:

$$\psi_2^\vee : \nu_M(\beta_Y(\mathcal{O}_X)) \longrightarrow ((\tau^*)_*(\mathcal{C}_{M_+}|X))^\vee[-n-1].$$

On the other hand for a section $f_1(\tilde{t},z)Y(t)+[f_2(\tilde{t},z)]\in\mathcal{O}_X(D_\varepsilon^+(\mathring{x},\hat{y}))\times Y(t)+H_Y^1(D_\varepsilon(\mathring{x},\hat{y}),\mathcal{O}_X)$ with a $f_2(\tilde{t},z)\in\mathcal{O}_X(D_\varepsilon(\mathring{x},\hat{y})\setminus Y)$ we can define a section

$$f_1(re^{i\theta},z)Y(r)+f_2((r+i0)e^{i\theta},z)-f_2((r-i0)e^{i\theta},z)$$

of $\Gamma(W_{\varepsilon'}(p_0),\beta_Y(\mathcal{O}_X))$ with a smaller $\varepsilon'>0$ (see (11)). It is easy to see that this gives a natural $\mathcal{D}_X$-morphism for sheaves:

$$\psi_1^{\vee} : (\overset{\circ}{C}_{M_+|Z})^{\vee}[-n-1] \longrightarrow \nu_M(\beta_Y(\mathcal{O}_X)),$$

and that $\psi_2^{\vee}\cdot\psi_1^{\vee}$ coincides with the natural $\mathcal{D}_X$-morphism. This completes the proof.

As a corollary of Proposition 11 we have the following:

<u>Theorem 12.</u> Let $P(t,x,D_t,D_x)$ be a C^{ω}-differential operator defined in a neighborhood of $(0,\overset{\circ}{x})$. We assume that $N=\{t=0\}$ is non-characteristic for P. Set a $\mathcal{D}_X$-module

$$M = \mathcal{D}_X/\mathcal{D}_X\tilde{t}^{\ell}P(\tilde{t},z,D_{\tilde{t}},D_z)$$

for an integer $\ell\geq 0$. Then we have the following isomorphisms for complexes

$$\mathbf{R}\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}M,\overset{\circ}{C}_{M_+|Z}) \overset{\sim}{\to} \mathbf{R}\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}M,C_{M_+|Z})$$

$$\overset{\sim}{\to} \mathbf{R}\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}M,(\tau^*)_*(C_{M_+|X})). \qquad (83)$$

In particular the microsupports of these complexes coincide with that of $SS(\mu_M(\mathbf{R}\mathcal{H}om_{\tau^{-1}\mathcal{D}_X}(\tau^{-1}M,\beta_Y(\mathcal{O}_X)))$.

<u>Proof.</u> It is sufficient to show that the isomorphism

$$\mathbf{R}\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}M,\overset{\circ}{C}_{M_+|Z}) \overset{\sim}{\to} \mathbf{R}\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}M,(\tau^*)_*(C_{M_+|X}))$$

on $\{r=0\}$. In particular we have only to show the isomorphism

$$\mathbf{R}\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}M,\mathrm{ext}(\overset{\circ}{C}_{N|M_+})+\rho^{-1}C_{N|Z})$$

$$\overset{\sim}{\longrightarrow} \mathbf{R}\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}M,\iota_*(C_{M_+|X}|_{T_N^*X})), \qquad (84)$$

where we used the same notation π for projections $\pi_{M/Z}$, $\pi_{N/M}$.

Indeed, noting that $\rho^{-1}\mathcal{C}_{N|Z} \subset \iota_*(\mathcal{C}_{N|X})$ we easily obtain the isomorphisms for the 0-th and the first cohomology groups in (84) from a division theorem on sections of $\mathcal{C}_{N|X}$ and $\mathcal{C}_{M_+|X}$. We omit the details.

Lemma 13. $(\tau^*)_*(\mathcal{C}_{M_+|X})$ is conically flabby on $T^*_M Z$. In particular, for any conically closed subset S of $T^*_M Z$ we have the quasi-isomorphism:

$$\Gamma_S((\tau^*)_*(\mathcal{C}_{M_+|X})) \xrightarrow{\sim} \mathbf{R}\Gamma_S((\tau^*)_*(\mathcal{C}_{M_+|X})). \tag{85}$$

Proof. Set $M_-=\{(t,x)\in M; t\le 0\}$. Then we have an exact sequence of sheaves on $T^*_M Z$:

$$0 \longrightarrow (\tau^*)_*(\mathcal{C}_{N|X}) \longrightarrow (\tau^*)_*(\mathcal{C}_{M_+|X})\oplus(\tau^*)_*(\mathcal{C}_{M_-|X})$$
$$\xrightarrow{\kappa} (\tau^*)_*(\mathcal{C}_M) \longrightarrow 0. \tag{86}$$

Indeed the surjectivity of κ comes from the surjectivity of the morphisms:

$$\Gamma_{M_+}(\mathcal{B}_M)\oplus\Gamma_{M_-}(\mathcal{B}_M) \longrightarrow \mathcal{B}_M \quad \text{and} \quad \pi^{-1}\mathcal{B}_M \longrightarrow (\tau^*)_*(\mathcal{C}_M).$$

Therefore the flabbiness of $(\tau^*)_*(\mathcal{C}_{M_\pm|X})$ reduces to that of $(\tau^*)_*(\mathcal{C}_{N|X})$ and $(\tau^*)_*(\mathcal{C}_M)$. Note here that $(\tau^*)_*(\mathcal{C}_{N|X})$ is isomorphic under a quantized contact transformation to a sheaf $q_*(\mathcal{C}_x\mathcal{O}_w)$. Here $\mathcal{C}_x\mathcal{O}_w$ is the sheaf on $iT^*\mathbf{R}^n_x\times\mathbf{C}_w$ of microfunctions with 1-holomorphic parameter w, and $q : iT^*\mathbf{R}^n_x\times\mathbf{C}_w \longrightarrow iT^*\mathbf{R}^n_x$ is the projection. On the other hand, we obtained in [6] the partial flabbiness of $\mathcal{C}_x\mathcal{O}_w$ with respect to $(x;i\eta dx)$-coordinates. In particular $q_*(\mathcal{C}_x\mathcal{O}_w)$, and so $(\tau^*)_*(\mathcal{C}_{N|X})$ are conically flabby.

Thus the proof is completed.

As a direct corollary from Theorem 12 and Lemma 13 we have a quasi-isomorphism:

$$\mathbf{R}\Gamma_S(\mu_M(\mathbf{R}\mathcal{H}om_{\tau^{-1}\mathcal{D}_X}(\tau^{-1}\mathcal{M}, \beta_Y(\mathcal{O}_X))))$$

$$\xrightarrow{\sim} \mathbf{R}\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{M}, \Gamma_S((\tau^*)_*(\mathcal{C}_{M_+}|X))) \qquad (87)$$

for any conically closed subset S of $T^*_M Z$ and for $\mathcal{M}$ in Theorem 12. Hence applying Theorem 12 to the operator P in Theorem 7, we have some theorems on unique continuation and existence for microlocal solutions.

<u>Theorem 14.</u> Let $P=D_t^2 + R(t,x,D_x)$ be a second order C^ω-differential operator defined in a neighborhood of $(0,\mathring{x})\in\mathbf{R}_t\times\mathbf{R}^n_x$ with real principal symbol. Let $\psi(r,x,\theta^*,\eta)$ be a real C^1-function defined in a neighborhood of $p_0=(0,\mathring{x};0d\theta-\mathring{\eta}dy)\in \mathring{T}^*_M Z$ such that ψ is homogeneous of degree 0 in (θ^*,η) with $d\psi(0,\mathring{x},0,\mathring{\eta})\neq 0$. We assume the following (i),(ii):

(i) $\sigma(R)(0,\mathring{x},i\mathring{\eta})=0$,

(ii) $(\partial_\eta\psi\cdot\partial_x-\partial_x\psi\cdot\partial_\eta)\sigma(R)(r,x,i\eta)|_{r=0,x=\mathring{x},\eta=\mathring{\eta}} \neq 0. \qquad (88)$

Then, we have a unique continuation property for $\mathring{\mathcal{C}}_{\Omega|Z}$-solution :

$$\left\{u\in(\mathring{\mathcal{C}}_{\Omega|Z})_{p_0};\ \begin{matrix}Pu=0\\ u(+0,x)=0\end{matrix}\right\} \xrightarrow{\sim} \left\{u\in\Gamma_{\{\psi<0\}}(\mathring{\mathcal{C}}_{\Omega|Z})_{p_0};\ \begin{matrix}Pu=0\\ u(+0,x)=0\end{matrix}\right\}.$$

Further, for any $f(t,x)\in\Gamma_{\{\psi\geq 0\}}(\mathring{\mathcal{C}}_{\Omega|Z})_{p_0}$ and any $g(x)\in\Gamma_{\{\psi_0'(x,\eta)\geq 0\}}(\mathcal{C}_N)_{p_0}$ with $\psi_0(x,\eta)=\psi(0,x,0,\eta)$ we have a unique solution $u(t,x)$

$\in\Gamma_{\{\psi\geq 0\}}(\mathring{C}_{\Omega|Z})_{p_0}$ to the equations:

$$\begin{cases} Pu(t,x) = f(t,x), \\ u(+0,x) = g(x). \end{cases} \tag{89}$$

Remark. For a hyperfunction solution u(t,x) of Pu=0 in {t>0}, we mean that

$$[u]\in\Gamma_{\{\psi\geq 0\}}(\mathring{C}_{\Omega|Z}) \longleftrightarrow \begin{cases} SS(u)\cap\{t>0\}\subset\{(t,x;i\eta_t dt+i\eta dx); \\ \qquad\qquad \psi(t,x,-t\eta_t,\eta)\geq 0\}, \\ \text{and } \bigcup_{j=0}^{1} SS(D_t^j u(+0,x))\subset\{\psi_0(x,\eta)\geq 0\}. \end{cases}$$

Proof. By Theorem 7 and the argument at (87) we have an exact sequence

$$0 \longrightarrow \Gamma_{\{\psi\geq 0\}}((\tau^*)_*(C_{M_+|X}))_{p_0} \xrightarrow{tP} \Gamma_{\{\psi\geq 0\}}((\tau^*)_*(C_{M_+|X}))_{p_0} \longrightarrow 0.$$

Then by the flabbiness of $(\tau^*)_*(C_{M_+|X})$ we can reduce this theorem to the exactness of the above sequence. We omit the details because the argument goes in the same way as in [2].

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